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LINEAR ORDER IN THREE DIMENSIONAL EUCLIDEAN AND DOUBLE ELLIPTIC SPACES.*

BY GEORGE H. HALLETT, JR.

1. **Introduction.** In this paper I shall deal with Euclidean and double elliptic geometries of three dimensions, the notions of point and order being undefined. A representation of double elliptic geometry of three dimensions is a geometry on the hypersphere

$$x^2 + y^2 + z^2 + w^2 = 1$$

in space of four dimensions. A plane in this geometry is the intersection of this hypersphere with a three-dimensional space through the origin

$$ax + by + cz + dw = 0.$$

The unit sphere in x, y, z-space

$$x^2 + y^2 + z^2 = 1, \qquad w = 0,$$

is a particular plane. A line is the intersection of two distinct planes. In particular all great circles on the above-mentioned unit sphere are lines. Complete treatments of three-dimensional Euclidean and double elliptic geometries in terms of point and order have already been given.†

The problem of this paper may be stated as follows. Two sets of postulates A and B are desired such that

1. Every proposition of Euclidean geometry of three dimensions can be deduced from the postulates of A.

2. Every proposition of double elliptic geometry of three dimensions can be deduced from the postulates of B.

3.‡ No postulates of either set can be derived from the other postulates of that set.

4. All the propositions true for a line in either geometry can be derived from postulates of the set corresponding to the geometry which do not necessitate the existence of a point off the line.

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* Presented to the Society, April 27, 1918.

† O. Veblen, "A System of Axioms for Geometry," Transactions of the American Mathematical Society, vol. 5 (1904), pp. 343–384.

J. R. Kline, "Double Elliptic Geometry in Terms of Point and Order Alone," Annals of Mathematics, vol. 18 (1916), pp. 31–44.

‡ I have not completely proved that my Double Elliptic sets satisfy Condition 3. Cf. footnote below concerning 7B.

5. The sets A and B have in common a set of postulates C.

6. The number of postulates of A and B which do not belong to C is as small as conveniently possible.

7. The postulates of C are natural for both geometries. For example no parts of postulates are vacuously satisfied in either geometry.

8. No postulate of A which does not belong to C is natural for double elliptic geometry, and no postulate of B which does not belong to C is natural for Euclidean geometry.

This problem was suggested by Professor Robert L. Moore, to whom I am likewise indebted for many valuable suggestions in its solution.

I give three pairs of sets of postulates, Σ_1–Σ_3. Use is made of notions of order somewhat different from those used by Veblen and Kline. Veblen's axioms are satisfied by a geometry of ordinary Euclidean space in which three points A, B and C are regarded as having the order ABC if they are all distinct and B lies between A and C on the straight line joining them. $\Sigma_1 A$, $\Sigma_2 A$ and $\Sigma_3 A$ are satisfied by a Euclidean geometry in which the additional orders AAB, ABB and AAA are true for every pair of points A and B. This notion is due to Kempe.* Kline's axioms are satisfied by a double elliptic geometry in which A, B and C are in the order ABC if B lies between A and C on an arc of less than 180° of one of the great circles which constitute the lines of the geometry.† All my double elliptic sets are satisfied by geometries in which the arc from A to C through B is $\leqq 180^\circ$. Two points which are the end points of an arc of 180° are called opposites. If (a, b, c, d) is a point on the hypersphere

$$x^2 + y^2 + z^2 + w^2 = 1,$$

$(-a, -b, -c, -d)$ is its opposite. Every point of the geometry is between any two such points with my revised notion of order.

When this proposition is established it becomes very powerful in the further development. In $\Sigma_1 B$, $\Sigma_2 B$ and $\Sigma_3 B$ the orders AAB, ABB and AAA are true for every pair of points A and B.

To show that Σ_1–Σ_3 satisfy conditions 1 and 2 it is sufficient to show that the equivalents of Veblen's axioms for the notions of order used can be deduced from each of the sets $\Sigma_1 A$, $\Sigma_2 A$, $\Sigma_3 A$, and that the equivalents of Kline's axioms can be deduced from each of the sets $\Sigma_1 B$, $\Sigma_2 B$, $\Sigma_3 B$. The categoricity of my sets then follows from the categoricity of Veblen's and Kline's sets.

To show that Condition 3 is satisfied by each set it is sufficient to give for each postulate of the set an independence example, a geometry in which that postulate is not satisfied but all other postulates of the set are satisfied.[*]

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* A. B. Kempe, "On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points," Proceedings of the London Mathematical Society, vol. 21 (1890), pp. 147–182.

† This notion was first introduced by Halsted in his Rational Geometry.

In this paper I shall make frequent use of Veblen's and Kline's papers. I shall not acknowledge each item separately.

2. **The Set Σ_1.** POSTULATE 1.[*] *There exist two distinct points.*

DEFINITION.[†] *B is between A and C* if *ABC* or *CBA*.

POSTULATE 2.[‡] *If A and B are any two distinct points, there exists a point C between A and B and distinct from both.*

POSTULATE 3.[§] *If ABA, $B = A$.*

POSTULATE 4.[*] *If ABC then CBA.*

POSTULATE 5. *If ABC and ACD then BCD.*

POSTULATE 6.[†] *If ABC and ABD, then either ACD or ADC or $A = B$.*

DEFINITIONS.[‡] If *C* is a point in the order *ACB* but not in either of he orders *CBA*, *BAC*, the *line ACB* is the set of all points in some order with *C* and either *A* or *B*. If *A* and *B* are two distinct points of a line *l*, the *segment AB* of *l* is the set of all points of *l* between *A* and *B* other than *A* and *B*, which are called the *end points* of the segment.

POSTULATE 7. SEPARABILITY. *If [S] is an infinite set of distinct points all on the same line, there exists a countable[*] set of points [K] on that line such that between every two distinct points of [S] there is a point of [K].*

POSTULATE 8. *If [T] is a countably infinite set of distinct segments contained in a segment AB such that (1) every point of AB is in infinitely many segments of [T] and (2) no two distinct points of AB are in infinitely*

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[*] In none of my double-elliptic sets have I succeeded in constructing such an example for Postulate 7*B*.

[†] This is Veblen's Axiom 1.

[‡] In this paper letters denote points unless otherwise stated. The abbreviation "*ABC*" will be used for the expression "points *A*, *B* and *C* are in the order *ABC*."

[§] In Veblen's and Kline's treatments this proposition can be proved by means of the two dimensional triangle axiom (Veblen's Axiom 8, Kline's Axiom 7, my Postulate 10), but cannot be proved on the basis of the one-dimensional axioms. Condition 4 is therefore not satisfied in their treatments. In Veblen's treatment the proof involves Axiom 5 (my Theorem 9*A*), in the proof of which Postulate 2 is needed in my treatment.

The expression $B = A$ means that *B* and *A* are the same point. $B \neq A$ means that *B* and *A* are distinct points.

[||] Cf. Veblen's Axiom 4.

[¶] The equivalents of Veblen's Axiom 2 and Kline's Axiom 3 are special cases of this postulate.

[**] Postulates 5 and 6 are used in place of Veblen's Axiom 6, which, if expressed in terms of point and order alone, is somewhat complicated.

[††] The definition is given thus in order that the same definition may serve in both geometries and that the mere existence of a point off a line may not imply anything about order on the line. In view of Theorems 3 and 4 and Postulate 4 the condition that *CBA* and *BAC* are false is equivalent in the development from this set of postulates to the condition $A \neq C \neq B$.

[‡‡] A *countable* set of points is one which can be brought into one-to-one correspondence with the set of all positive integers or a part of that set. All the points of [*K*] may thus be represented by the sequence $K_1, K_2, K_3, \cdots$.

many segments of $[T]$, *then if* $[T']$ *consists of an infinite number of the segments of* $[T]$, *there exists a point* P *between* A *and* B *such that* (1) *if* $A \neq P \neq B$ *every sub-segment of* AB *containing* P *contains a segment of* $[T']$, *and* (2) *if* $P = A$ (*or* B) *every sub-segment of* AB *determined by* A (*or* B) *and a point of the segment* AB *contains a segment of* $[T']$.

In place of Postulates 7 and 8 Veblen uses the Heine-Borel Theorem (Axiom 11) and Kline uses the Dedekind Cut Postulate (Axiom 10). If it were not for the requirements of Condition 4 either of these propositions might be used here with slight modifications instead of Postulate 8. Examples 7 would serve as independence examples for either. But in his paper entitled Definition in Terms of Order Alone in the Linear Continuum and in Well-ordered Sets (Transactions of the American Mathematical Society, vol. 6, 1905, pp. 165–171) Veblen gives an example for his uniformity postulate with the aid of which it may be seen that separability would not follow from the linear postulates of the set obtained from Σ_1 by replacing Postulate 8 by the Heine-Borel or Dedekind Cut Postulate. Moreover separability could not be assumed here in addition to the Dedekind Cut Postulate without destroying the independence of the set. For Professor Robert L. Moore has shown that separability follows from the Dedekind Cut Postulate and the order postulates with the aid of the two-dimensional triangle postulate (my Postulate 10). This was done in his paper On a Set of Postulates which Suffice to Define a Number Plane, Transactions of the American Mathematical Society, vol. 16 (1915), pp. 27–32. Separability could not be assumed in addition to the Heine-Borel Theorem for a similar reason. The difficulty is obviated by assuming Postulate 8 in addition to separability and proving the Dedekind Cut Postulate and the Heine-Borel Theorem with the aid of these two linear postulates. This solution of the problem is due to Professor Moore.

POSTULATE 9. *No line contains all points.*

POSTULATE 10. *If* ABC *and* BDE *are two lines and* E *is not on the line* ABC, *there exists a point* F *between* A *and* E *and on a line containing* C *and* D.

DEFINITIONS. If $A \neq B$ and C is not on a line containing A and B, the *triangle* ABC consists of all points between two of the points A, B, C. The *plane* ABC consists of all points of all lines containing two distinct points of the triangle ABC.

POSTULATE 11. *No plane contains all points.*

DEFINITIONS. If A, B and C determine a plane and D is not in that plane, the *tetrahedron* $ABCD$ consists of all points which are between two points of a triangle determined by three of the points A, B, C, D.

The *space ABCD* consists of all points of all lines which contain two distinct points of the tetrahedron $ABCD$.

POSTULATE 12. *If four points determine a space, there is no point which does not belong to that space.*

POSTULATE 13A.[*] *If A, B, C and D are four distinct points, then there exists a point E in the order ABE and not between C and D.*

POSTULATE 14A. *If a is any line of any plane α and there is a point of α not on a, then there is some point of α not on a through which there is not more than one line of α which contains no point of a.*

POSTULATE 13B. *If A, B and C are any three points and there exist points X and Y in the orders ABX and ACY,*[†] *then there exists a point D in the orders ABD and ACD.*

INDEPENDENCE EXAMPLES.

EXAMPLE 1.[‡] No points. No order.

EXAMPLE 2. Two points. No order.

EXAMPLE 3. Three distinct points. All possible orders.

EXAMPLE 4A. Euclidean geometry[§] with the following modification. There is a unique line l through a unique point A. BCA is never true if B and C lie on l to the left of A and A, B and C are all distinct. All other usual orders are true.

EXAMPLE 4B. Double elliptic geometry with the modification that on a certain line there are no clockwise orders except those involving opposites or two identical points.

EXAMPLES 5A AND 5B. Euclidean and double elliptic geometries with the modification that AAA is never true.

EXAMPLES 6A AND 6B. Euclidean and double elliptic geometries with the convention that ABC implies that A, B and C are all distinct.

EXAMPLE 7A. $[G]$ is a well-ordered set of symbols in one-to-one correspondence with the set of all points on a line. All rational functions of the symbols of $[G]$ are called numbers. Two such numbers are equal if one can be reduced to the other by ordinary algebraic operations. To compare two polynomials in the symbols of $[G]$ proceed as follows. Let $G_1, G_2, \cdots G_n$ be the symbols of $[G]$ involved so arranged that G_i follows

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[*] This modification of Veblen's Axiom 5 is useful not merely in satisfying Condition 4, but also in proving another four-point proposition (Theorem 11) from Postulates 5 and 6.

[†] This condition is imposed in order that Postulate 13B may not imply Theorem 3 which is easily proved from postulates of the common basis.

[‡] Independence examples are numbered to correspond to the postulates whose independence they prove. When, as in this case, the same example will serve in both systems, no letter is used. Otherwise the example in $\Sigma_1 A$ is marked A and the example in $\Sigma_1 B$ is marked B.

[§] "Euclidean geometry" or "double elliptic geometry" will mean in this paper the geometry given by the full set of postulates under consideration.

G_{i+1} for $i = 1, 2, \cdots n - 1$. Regard $G_2, \cdots G_n$ as positive constants and allow G_1 to increase indefinitely. If as G_1 increases one polynomial becomes and remains greater than the other irrespective of the values assigned to $G_2, \cdots G_n$, it is greater. Otherwise regard $G_1, G_3, G_4, \cdots G_n$ as positive constants and allow G_2 to increase indefinitely. If one polynomial thus becomes and remains greater than the other, it is greater. Otherwise continue this process as many times as may be necessary to reach a decision. A decision will always be reached unless the two polynomials are equal. A positive polynomial is one greater than zero. To compare two fractional functions of the symbols of $[G]$, reduce to a common positive denominator and compare numerators. The number system thus ordered satisfies Postulates 1–6. It is not separable and the Dedekind Cut Postulate fails. Postulate 8 is vacuously satisfied.

On this number system a three dimensional geometry similar to Euclidean geometry may be built as follows. A point is a set of three numbers (a, b, c). A point (a, b, c) satisfies an equation

$$f(x, y, z) = 0$$

if the equation is satisfied when the substitutions $x = a$, $y = b$, $z = c$ are made. The set of all points satisfying a particular linear equation

$$c_1x + c_2y + c_3z + c_4 = 0$$

is called a plane. A line is the set of all points common to two distinct planes. Points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ have the order ABC if $A = B$ or $B = C$, or one of the sets of inequalities

$$x_1 > x_2 > x_3, \qquad x_1 < x_2 < x_3,$$

$$y_1 > y_2 > y_3, \qquad y_1 < y_2 < y_3,$$

$$z_1 > z_2 > z_3, \qquad z_1 < z_2 < z_3,$$

is true and A, B and C are on the same line.

EXAMPLES 8*A* AND 8*B*. Rational Euclidean and double elliptic geometries.

EXAMPLE 9*A*. One-dimensional Euclidean geometry. Geometry on a line.

EXAMPLE 9*B*. One-dimensional double elliptic geometry. Geometry on a circle.

EXAMPLE 10*A*. Euclidean geometry with the following modification. There are two distinct peculiar points, A and B, A is substituted for B and B for A in any order involving at least one point off the line AB.

EXAMPLE 10*B*. Two planes divide the remainder of double elliptic space into two pairs of precisely similar divisions. The pairs are not similar unless the planes are orthogonal. All points of one pair are retained, all points of the other discarded. All points of one bounding plane are retained, all points of the other which are not points of the first are discarded. Order among the remaining points is the same as in the full double elliptic system.

Example 11A. Two-dimensional Euclidean geometry. Geometry on a plane.

Example 11B. Two-dimensional double elliptic geometry. Geometry on a sphere.

Examples 12A and 12B. Euclidean and double elliptic geometries of four dimensions.

Example 13A. Double elliptic geometry.

Example 14A.[*] Bolyai-Lobatchewskian geometry.

Example 13B. Euclidean geometry.

Theorems derived from the common basis.

Theorem 1. *If* $A \neq B$, ABB.

Proof. There exists C in the order ACB such that $A \neq C$ (p.[†] 2 and p. 4). Since ACB and ACB and $A \neq C$, ABB (p. 6).

Theorem 2. *If* A *is any point*, AAA.

Proof. There exists B distinct from A (p. 1). BAA (th. 1). $\therefore AAA$ (p. 5).

Theorem 3. If A and B are any two points, ABB and AAB.

Proof. ABB and BAA (ths. 1 and 2). $\therefore AAB$ (p. 4).

Theorem 4. *If* ABC *and* ACB, $B = C$.

Proof. BCB (p. 5). $\therefore B = C$ (p. 3).

Corollary.[‡] *If* ABC, BCA *is not true unless* $B = C$.

Theorem 5.[§] *There exist three distinct points in the order* ABC.

Proof. There exist two distinct points, A and C (p. 1). There exists B distinct from both in the order ABC (p. 2 and p. 4).

Theorem 6. *If* ABC *and* ACD, ABD.

Proof. BCD (p. 5). DCA and DCB (p. 4). $\therefore DAB$ or DBA or $D = C$ (p. 6). If $D = C$, ABC means ABD. If DAB, DCA gives CAB (p. 5). Since CBA (p. 4), $A = B$ (th. 4). $\therefore ABD$ (th. 3). If DBA, ABD (p. 4). Thus in any case, ABD.

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[*] In view of this example the common basis here given will serve as a common basis for Euclidean, Bolyai-Lobatchewskian and double elliptic geometries, and the common basis together with Postulate 13A will serve as a common basis for Euclidean and Bolyai-Lobatchewskian geometries.

[†] In proof references, p. = Postulate, th. = Theorem.

[‡] Cf. Veblen's Axiom 3.

[§] This is the equivalent of Kline's Axiom 2.

THEOREM 7.* *If ABC and ACD and* $B \neq C$, $B \neq D$.

Proof. If $D = B$, ACB. But ABC. $\therefore B = C$ (th. 4), contrary to hypothesis.

THEOREM 8. *If ABC and ABD, BCD or BDC or* $A = B$.

Proof. ACD or ADC or $A = B$ (p. 6). If ACD, ABC gives BCD (p. 5). If ADC, ABD gives BDC (p. 5).

Theorem 4 may be substituted in Σ_1 for Postulate 3. If this is done the same independence examples will serve and the development proceeds as follows.

THEOREMS 1–3 as before.

POSTULATE 3′. *If ABC and ACB,* $B = C$.

THEOREM 4′. *If ABA,* $B = A$.

Proof. If $A \neq B$ then AAB (th. 1 and p. 4). But ABA. $\therefore B = A$ (p. 3′).

The development then continues as before. Postulates 3 and 3′ are thus seen to be equivalent in the presence of Postulates 1, 2, 4–12, 13*A* and 14*A* or Postulates 1, 2, 4–12 and 13*B*, or in fact Postulates 2, 4, 5 and 6, the only other postulates used in proving Postulate 3′ from Postulate 3 and Postulate 3 from Postulate 3′.

EUCLIDEAN GEOMETRY FROM $\Sigma_1 A$.

THEOREM 9.† *If A and B are any two points, there exists a point C distinct from B in the order ABC.*

Proof. Case 1. $A = B$. There exists C distinct from B (p. 1). ABC (th. 3).

Case 2. $A \neq B$. There exist points D and E such that $ADB[A \neq D \neq B]$ and $AED[A \neq E \neq D]$ (p. 2 and p. 4). $B \neq E$, for otherwise $B = D$ follows from ADB and ABD (th. 4). $\therefore A, B, D$ and E are all distinct. There exists therefore a point C in the order ADC and not between E and B (p. 13A). AED and ADC. $\therefore AEC$ (th. 6). ADC and ADB and $A \neq D$. $\therefore ACB$ or ABC (p. 6). If ACB, AEC gives ECB (p. 5), contrary to hypothesis. $\therefore ABC$. Since C is not between E and B, $C \neq B$ (th. 3).

THEOREM 10. *If ABC and ADC, ABD or ADB.*

Proof. There exists E distinct from A in the order CAE (th. 9). CBA and CDA (p. 4). $\therefore EAB$ and EAD (p. 5 and p. 4). $\therefore ABD$ or ADB (th. 8).

THEOREM 11. *If ABC and BCD then either ABD or* $B = C$.

Proof Case 1. $A = B$. ABD (th. 3).

Case 2. $A = C$. CBC. $B = C$ (p. 3).

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* Cf. Theorems 6 and 7 with Kline's Axiom 5.

† Cf. Theorem 9, Case 2 with Veblen's Axiom 5.

Case 3. $A = D$. ABC and ACB (p. 4). $B = C$ (th. 4).

Case 4. $B = D$. BCB. $B = C$ (p. 3).

Case 5. $C = D$. ABD.

Case 6. A, B, C and D all distinct. There exists E in the order ACE and not between B and D. ABC and ACE. $\therefore BCE$ (p. 5) and ABE (th. 6). BCD. $\therefore BED$ or BDE (p. 6). But E is not between B and D by hypothesis. $\therefore BDE$. EDB and EBA (p. 4). $\therefore ABD$ (p. 5 and p. 4[*]).

THEOREM 12.[†] *If A and B are any two distinct points there is just one line ACB and a necessary and sufficient condition that a point X be on this line is that it be in one of the orders XAB, AXB, ABX.*

Proof. Let C be any point such that ACB, but not CBA or BAC. Then ACB is a line. C is distinct from A and B (th. 3). Let X be any point on the line ACB.

Case 1. XAC. ACB. $\therefore XAB$ (th. 11).

Case 2. AXC. ACB. $\therefore AXB$ (th. 6).

Case 3. ACX. ACB. $\therefore CXB$ or CBX (th. 8). See Cases 5 and 6.

Case 4. XCB. ACB. $\therefore XAC$ or AXC (th. 8). See Cases 1 and 2.

Case 5. CXB. ACB. $\therefore AXB$ (th. 6).

Case 6. CBX. ACB. $\therefore ABX$ (th. 11).

The necessity of the condition is thus established.

Now let X be any point in one of the orders XAB, AXB, ABX, and C as before any point such that ACB, but not CBA or BAC.

Case 1. XAB. ACB. $\therefore XAC$ (p. 5).

Case 2. AXB. ACB. $\therefore AXC$ or ACX (th. 10).

Case 3. ABX. ACB. $\therefore CBX$ (p. 5).

The sufficiency of the condition is thus also established. As the condition stated in the theorem does not involve C and has been proved equivalent to the definition of a line, it follows that there is not more than one line ACB. As there is always a point C (p. 2 and th. 4) and a point in some order with A and C (th. 3), one line always exists.

THEOREM 13.[‡] *If C and D are any two distinct points of the line AB, A is a point of the line CD.*

Proof. C is in one of the orders $(\alpha)CAB$, $(\beta)ACB$, $(\gamma)ABC$ (th. 12). D is in one of the orders $(a)DAB$, $(b)ADB$, $(c)ABD$.

Case αa. BAC and BAD. $\therefore ACD$ or CDA (th. 8).

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[*] Hereafter Postulate 4 will be used without reference.

[†] This theorem establishes the equivalence of Veblen's and my definitions for a line. In view of this equivalence the equivalents of Veblen's Axioms 7, 9 and 10 follow from my Postulates 9, 11 and 12.

[‡] This theorem is the equivalent of Veblen's Axiom 6 in view of Theorem 12. The proof of this theorem illustrates the advantage of the Kempe notion of order. If Veblen's order were used, 25 cases would have to be considered instead of 9.

Case αb. BAC and BDA. $\therefore CAD$ (p. 5).
Case αc. CAB and ABD. $\therefore CAD$ (th. 11).
Case βa. BCA and BAD. $\therefore CAD$ (p. 5).
Case βb. ACB and ADB. $\therefore ACD$ or CDA (th. 10).
Case βc. ACB and ABD. $\therefore ACD$ (th. 6).
Case γa. CBA and BAD. $\therefore CAD$ (th. 11).
Case γb. ABC and ADB. $\therefore CDA$ (th. 6).
Case γc. ABC and ABD. $\therefore ACD$ or CDA (p. 6).

Thus in any case A is on the line CD (th. 12).

THEOREM 14. *If C and D are distinct points of the line AB, the line CD is the line AB.*

Proof. Suppose $C \neq A$. Then the lines AC and AB are identical. For let X be any point of AB. If $X = C$, AXC (th. 1) and X is on AC. If $X \neq C$, A is on CX (th. 13). $\therefore AXC$, XAC or XCA. $\therefore X$ is on AC. Similarly, since B is a point of AC (th. 13), every point of AC is a point of AB. The lines AC and CD are similarly identical. Therefore the lines AB and CD are identical.

If $C = A$, $D \neq A$. By reasoning similar to the above,

$$AB = AD = CD.$$

COROLLARY. *Two distinct lines cannot have more than one point in common.*

For the sake of brevity I shall assume in the remainder of the development certain properties of linear order which follow easily from those already established.

THEOREM 15.[*] *If $ABC[A \neq B \neq C]$ and $BDE[B \neq D \neq E]$ and E is not on the line, AB then $C \neq D$ and there exists a point F on the line CD such that $EFA[E \neq F \neq A]$.*

Proof. Line ABC = line AC (th. 12). B is on AC and distinct from A. $\therefore ABC = AC = AB$ (th. 14).

$C \neq D$. For if $C = D$, AB and EB have in common two distinct points B and C, and are therefore identical (th. 14). But E is on EB and not on AB by hypothesis.

There exists a point F on the line CD between A and E (p. 10).

$A \neq F$. For if $A = F$, AB and CD have in common two distinct points A and C and are therefore identical. The lines AB and BE have only B in common. Since D is on BE and is distinct from B, D is not on AB. But D is on CD. $\therefore AB \neq CD$.

$E \neq F$. For if $E = F$, CD and BE have in common the two distinct points D and E. $\therefore CD = BE$. BE has only B in common with AB. But CD has C in common with AB. $\therefore CD \neq BE$.

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[*] Cf. Veblen's Axiom 8.

THEOREM 16. DEDEKIND CUT POSTULATE. *If all the points of a segment AB are divided into two sets $[S_1]$ and $[S_2]$ such that no point of either is between two points of the other, there exists a point H between every point of $[S_1]$ and every point of $[S_2]$.*

Proof. There exists a countable set of points $[K]$ on the segment AB such that between every two distinct points of AB there is a point of $[K]$ (p. 7). The set of all segments determined by pairs of points of $[K]$ is countable. Let this set be A_1B_1, A_2B_2, A_3B_3, $\cdots$. Let A, C_{i1}, C_{i2}, $\cdots$ $C_{im_i}B$ be for every integer i the set of points A, B, A_1, B_1, A_2, B_2, $\cdots$ A_i, B_i arranged in the order in which they occur on the line AB, each point being counted only once. A countable set of segments $[T]$ is determined as follows. $T_1 = AC_{12}$, $T_2 = AC_{11}$, $T_3 = C_{11}B$, $T_4 = C_{11}C_{12}$, $T_5 = T_{l_1}{}^* = C_{12}B$, $T_{l_1+1} = AC_{22}$, $T_{l_1+2} = AC_{21}$, $T_{l_1+3} = C_{21}C_{23}$, $T_{l_1+4} = C_{21}C_{22}$, $\cdots$ $T_{l_1+l_2} = C_{2m_2}B$, $T_{l_1+l_2+1} = AC_{32}$, $T_{l_1+l_2+2} = AC_{31}$, $T_{l_1+l_2+3} = C_{31}C_{33}$, etc. One of the segments T_1, T_2, $\cdots$ T_5 has one end point in $[S_1]$, the other in $[S_2]$. Call the T of lowest subscript having this property T_1'. Similarly one of the segments $T_{l_1+\cdots+l_i+1}$, $T_{l_1+\cdots+l_i+2}$, $\cdots$ $T_{l_i+\cdots+l_i+l_{i+1}}$ has one end point in $[S_1]$ and the other in $[S_2]$, and the segment of lowest subscript in this set which has this property is called T_{i+1}'.

Every point of AB is contained in infinitely many of the segments of $[T]$, for every point of AB is contained in one of the segments $T_{l_1+\cdots+l_i+1}$, $T_{l_1+\cdots+l_i+2}$, $\cdots$ $T_{l_1+\cdots+l_i+l+l_{i+1}}$ for every integer i.

No two distinct points P and Q of AB are contained in infinitely many of the segments of $[T]$. For there exist distinct points K_1, K_2 and K_3 of $[K]$ in the order $PK_1K_2K_3Q$. Either the segment K_1K_2 or one included in it is a segment T_j of $[T]$. No segment of $[T]$ following T_j in the sequence T_1, T_2, T_3, $\cdots$ contains both P and Q.

There exists a point H between A and B such that (1) if $A \neq H \neq B$, every segment of AB containing H contains a segment of $[T'] = T_1'$, T_2', $\cdots$, and (2) if $H = A$ (or B) every segment of AB having H as one end point contains a segment of $[T']$ (p. 8). Either AS_1S_2B or AS_2S_1B is true for all points of $[S_1]$ and all points of $[S_2]$. We may assume without loss of generality that AS_1S_2B is always true. Suppose $A \neq H \neq B$. Then H belongs to either $[S_1]$ or $[S_2]$. If it belongs to $[S_1]$ there is no point of $[S_1]$ distinct from H in the order AHS_1B, for the segment AS_1 in that case would contain H but no segment of $[T']$. Similarly if H is a point of $[S_2]$, there is no point of $[S_2]$ distinct from H in the order AS_2HB. Therefore H is between every point of $[S_1]$ and every point of $[S_2]$ unless $H = A$ or B. But $H \neq A$, for otherwise, if S_1 is any point of $[S_1]$, the segment AS_1 contains no segment of $[T']$. Similarly $H \neq B$.

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[*] Here $l_1 = 5$.

It has been shown by Veblen that the Heine-Borel Theorem, Veblen's Axiom 11, follows from the Dedekind Cut Postulate and certain other properties of linear order which have been established.*

The equivalents of all of Veblen's axioms have now been proved on the basis of $\Sigma_1 A$. $\Sigma_1 A$ is therefore categorical.

DOUBLE ELLIPTIC GEOMETRY FROM $\Sigma_1 B$.

THEOREM 9. *Every point is between two points distinct from it.*

Proof. Let A be any point. There exists D distinct from A (p. 1), and B such that $ABD[A \neq B \neq D]$ (p. 2). There exists C such that BAC and BDC (th. 1 and p. 13B). If $C = A$, ADB. But ABD. $\therefore B = D$ (th. 4), contrary to hypothesis. $\therefore A$ is between B and C and distinct from both of them.

DEFINITION. A' is said to be an opposite of A if there is no point B distinct from A' in the order $AA'B$.

THEOREM 10.† *Every point has an opposite.*

Proof. Let A be any point. A is between two points C and D distinct from it (th. 9). There exists a point A' in the order ACA' and ADA'(p. 13B). Suppose B is a point distinct from A' in the order $AA'B$. Since ACA' and ADA', $CA'B$ and $DA'B$ (p. 5). $\therefore A'CD$ or $A'DC$ (th. 8). Since $A'CA$ and $A'DA$, CDA or DCA (p. 5). Since CAD, $D = A$ or $C = A$ (th. 4), contrary to hypothesis. Therefore there is no point B distinct from A$'$ in the order $AA'B$, and A' is an opposite of A.

COROLLARY. *If $ABC[A \neq B \neq C]$, BAD and BCD, then D is an opposite of B.*

THEOREM 11. *No point has more than one opposite.*

Proof. Suppose A' and A'' are opposites of A. There exists a point X in the orders $AA'X$ and $AA''X$ (p. 13B). By the definition of an opposite, $X = A' = A''$.

COROLLARY. *If A' is the opposite of A, $A' \neq A$.*

THEOREM 12. *If A' is the opposite of A, every point is between A and A'.*

Proof. Let P be any point. If $P = A$ or A', APA' (th. 3). Suppose $A \neq P \neq A'$. There exists a point Q in the orders PAQ and $PA'Q$ (p. 13B). $Q \neq A$, for otherwise $AA'P$ and $P = A'$ (by definition), contrary to hypothesis. There exists a point A'' in the orders APA'' and AQA'' (p. 13B), and A'' is an opposite of A (th. 10, cor.). $\therefore A'' = A'$ (th. 11). $\therefore APA'$.

THEOREM 13. *If A' is the opposite of A, A is the opposite of A'.*

Proof. If A is not the opposite of A', there exists B such that $A'AB$

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* Cf. O. Veblen, "The Heine-Borel Theorem," Bulletin of the American Mathematical Society, vol. 10 (1903–04), pp. 436–439.

† Cf. Kline's Axiom 1.

and $B \neq A$. But $A'BA$ (th. 12). $\therefore A = B$ (th. 4), contrary to hypothesis.

THEOREM 14.* *If ABC, then $A'CB$, $B'A'C$ and $A'B'C'$.*

Proof. ACA' (th. 12). $\therefore BCA'$ (p. 5). Similarly $B'A'C$ and $A'B'C'$.

THEOREM 15. *If ABC and ADC, then ABD or ADB or $C = A'$.*

Proof. $A'CB$ and $A'CD$ (th. 14). $\therefore A'DB$ or $A'BD$ or $C = A'$ (p. 6). $\therefore ABD$ or ADB (th. 14) or $C = A'$.

THEOREM 16. *If ABC and BCD, then ABD or $AB'D$ or $B = C$.*

Proof. BCA' (th. 14) and BCD. $\therefore A'DB$ or $DA'B$ or $B = C$ (p. 6). $\therefore ABD$ or $AB'D$ (th. 14) or $B = C$.

THEOREM 17.† *If A and B are any two distinct non-opposite points, there is just one line ACB and a necessary and sufficient condition that a point X be on this line is that either XAB, AXB, ABX or $AX'B$.*

Proof. Let C be any point distinct from A and from B such that ACB. By Theorem 4 neither CBA nor BAC. Then ACB is a line. Let X be any point on the line ACB.

Case 1. XAC. ACB. $\therefore XAB$ or $XA'B$ (th. 16). If $XA'B$, $AX'B$ (th. 14).

Cases 2–5 are the same as Cases 2–5 respectively in the proof of Theorem 12A.

Case 6. CBX. ACB. $\therefore ABX$ or $AB'X$ (th. 16). If $AB'X$, $AX'B$ (th. 14).

The necessity of the condition is thus established.

Now let X be any point such that either XAB, AXB, ABX or $AX'B$ and let C be selected as above.

Cases 1–3 are the same as Cases 1–3 in the proof of Theorem 12A, except that Theorem 15B replaces Theorem 10A in Case 2.

Case 4. $AX'B$. ACB. $AX'C$ or ACX' (th. 15). If $AX'C$, $X'CB$ (p. 5) and CBX (th. 14). If ACX', XAC (th. 14).

The sufficiency of the condition is thus established also. It follows (cf. proof of Theorem 12A) that any two distinct non-opposite points determine just one line.

THEOREM 18.‡ *If C and D are any two distinct non-opposite points of the line AB§, A is a point of the line CD.*

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* Cf. Kline's Axiom 4.

† This theorem establishes the equivalence of Kline's and my definitions for a line. In view of this the equivalents of Kline's Axioms 6, 8 and 9 follow directly from Postulates 9, 11 and 12 respectively.

‡ Here again the Kempe notion of order is of great advantage. Kline had 64 cases to consider instead of 16. If he had adopted the notion that every point is between any two opposites, but not the Kempe notion, he would have had 36 cases.

§ "The line AB" implies that A and B are distinct and not opposites, so that they determine a single line.

Proof. One of the orders $(\alpha)CAB$, $(\beta)ACB$, $(\gamma)ABC$, $(\delta)AC'B$ is true, and one of the orders (a)DAB, (b)ADB, (c)ABD, (d)$AD'B$ is true.

Cases αa–c, βa–c, γa–c are similar to the same cases in the proof of Theorem 13A.

Case αd. CAB and $AD'B$. $\therefore CAD'$ (p. 5). ACD (th. 14).

Case βd. ACB and $AD'B$. $\therefore ACD'$ or $AD'C$ (th. 15). CAD or $CA'D$ (th. 14).

Case γd. ABC and $AD'B$. $\therefore AD'C$ (th. 6). $CA'D$ (th. 14).

Case δa. $AC'B$ and DAB. $\therefore DAC'$ (p. 5). CDA (th. 14).

Case δb. $AC'B$ and ADB. $\therefore AC'D$ or ADC' (th. 15). $CA'D$ or CAD (th. 14).

Case δc. $AC'B$ and ABD. $\therefore AC'D$ (th. 6). $CA'D$ (th. 14).

Case δd. $AC'B$ and $AD'B$. $\therefore AC'D'$ or $AD'C'$ (th. 15). CDA or ACD (th. 14).

THEOREM 19. *If* C *and* D *are distinct non-opposite points of the line* AB, *the line* CD *is the line* AB.

The proof is similar to the proof of Theorem 14A.

THEOREM 20. *Two distinct lines have at most a point and its opposite in common.*

Proof. If the lines are lines determined by pairs of non-opposite points, the theorem follows immediately from Theorem 19. Consider the line $ABA'[A \neq B \neq A']$. Any point X on this line is in one of the orders XAB, AXB, ABX, $XA'B$, $A'XB$, $A'BX$. $\therefore XAB$, AXB, ABX or $AX'B$ (th. 14). $\therefore$ line $ABA' =$ line AB. Therefore all cases reduce to the case first considered.

THEOREM 21.[*] *If* $ABC[A \neq B \neq C \neq A']$ *and* $BDE[B \neq D \neq E \neq B']$ *and* E *is not on the line* AB, *there exists a point* F *such that* $CDF[C \neq D \neq F \neq C']$ *and* $EFA[E \neq F \neq A \neq E']$.

Proof. By reasoning similar to the proof of Theorem 15A, there exists F on the line CD $[C \neq D \neq C']$ and such that $EFA[E \neq F \neq A]$. $A \neq E'$ since E is not on the line AB.

$F \neq C$, for otherwise $AB = AE$ (th. 20), contrary to the hypothesis that E is not on AB. Similarly $F \neq C'$.

$F \neq D$, for otherwise $AE = BE$ (th. 20). $AB \neq BE$ since E is not on AB. Therefore AB and BE have only B and B' in common (th. 20). But if $AE = BE$, they must have A in common also. Similarly $F \neq D'$.

Since F is on CD, one of the orders FCD, CFD, CDF, $CF'D$ is true.

Suppose FCD. AFE. D is not on AE, by the reasoning used to prove that $F \neq D$. Therefore there exists X between E and D and on the line AC (p. 10). Since the lines ED and AC have only the points

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[*] Cf. Kline's Axiom 7.

B and B' in common (th. 20), $X = B$ or B'. $\therefore$ EBD or $EB'D$. But EDB and DEB' (th. 14). $\therefore$ $B = D$ or $E = B'$ (th. 4), contrary to hypothesis.

Suppose CFD. Then FDC' (th. 14). AFE. Since $AC' = AB$ and E is not on AB, C' is not on AE (th. 18). Therefore there exists X on ED and between A and C' (p. 10). Since AC' and ED have only B and B' in common, $X = B$ or B'. $\therefore$ ABC' or $AB'C'$. $\therefore$ CAB or ACB (th. 14). But CBA. $\therefore$ $A = B$ or $B = C$ (th. 4), contrary to hypothesis.

Suppose $CF'D$. Then $FC'D$ (th. 14). AFE. D is not on AE. Therefore there exists X between D and E and on the line $AC' = AB$ (p. 10). Since DE and AB have only B and B' in common, $X = B$ or B'. $\therefore$ DBE or $DB'E$. But EDB and DEB' (th. 14). $\therefore$ $B = D$ or $E = B'$, contrary to hypothesis. Therefore CDF.

The proof of the equivalent of Kline's Axiom 10 is similar to the proof of Theorem 16A.

The equivalents of all of Kline's axioms have now been established on the basis of $\Sigma_1 B$, which is therefore categorical.

3. **The Set Σ_2.** Σ_2 is a set in which Postulate 4 of Σ_1 is replaced by a third four-point postulate. Postulates 1–3, 5, 7–12, 13A, 14A, 13B are the same as the corresponding postulates in Σ_1 except that the first definition is modified as follows.

DEFINITION. B is *between* A and C if ABC.

POSTULATE 4. *If ABC and BDC, CDA.*

POSTULATE 6. *If ABC and DBC, ADC or DAC or $B = C$.*

INDEPENDENCE EXAMPLES.

All examples are those given under Σ_1 except the following.

EXAMPLES 4A AND 4B. Euclidean and double elliptic geometries with the modification that ABB is never true.

DEVELOPMENT.

THEOREM 1. *If $A \neq B$, AAB.*

Proof. There exists C distinct from B in the order ACB (p. 2). Since ACB and ACB and $C \neq B$, AAB (p. 6).

THEOREM 2.* *If ABC, CBA.*

Proof. If $C = A$, $B = A$ (p. 3). CBA means ABC, which is true by hypothesis.

If $C \neq A$, AAC (th. 1). Also ABC. $\therefore$ CBA (p. 4).

THEOREM 3.† *If ABC and ABD, ACD or ADC or $A = B$.*

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* Postulate $\Sigma_1 4$.

† Postulate $\Sigma_1 6$.

Proof. CBA and DBA (th. 2). $\therefore CDA$ or DCA or $A = B$ (p. 6). $\therefore ACD$ or ADC or $A = B$ (th. 2).

Σ_1 has now been proved on the basis of Σ_2, which is therefore categorical.

As before Postulate 3 may be replaced by Postulate 3′. Postulate 3 may be proved from Postulate 3′, with the help of Postulates 2 and 6, as follows.

If ABA, $B = A$.

Proof. If $A \neq B$, AAB (th. 1). ABA by hypothesis. $\therefore B = A$ (p. 3′), contrary to supposition.

4. **The Set Σ_3.** Although Postulate $\Sigma_1 10$ has been so phrased as to apply only in space of two or more dimensions, it applies in one dimension also if stated simply as follows.

If ABC and BDE, there exists a point F in the orders CDF and AFE.

This is due of course to the revised notion of order used. The same is true of Veblen's Theorem 13, which is used as a postulate in this set.

POSTULATES 1–3, 6–10 AND 11A are the same as Postulates 1–3, 7–9, 11, 12 and 13A respectively of Σ_2, except that the definition of a plane is modified.

POSTULATE 4. *If ABC and DBC, ADC or DAC or $B = C$.*

POSTULATE 5. *If ABC and ADE, there exists a point F in the orders BFE and CFD.*

DEFINITION. If $A \neq B$, and C is not on a line containing A and B, the *plane ABC* consists of all points of all lines containing two distinct points in some order with two of the points A, B, C.

POSTULATE 12A.* *If l is a line and C is any point, there is not more than one line in a plane containing l and C and containing C but no point of l.*

POSTULATE 11B. *If A, B and C are distinct points, there exists a point D in the orders ABD and ACD.*

INDEPENDENCE EXAMPLES.

EXAMPLES 1–3, 6, 7A, 8–10, 11A, 12A, 11B are the same as the examples for the corresponding postulates of Σ_1.

EXAMPLE 4A. Euclidean geometry with the modification that ABC implies that A, B and C are all distinct or all identical.

EXAMPLE 4B. Double elliptic geometry with the modification that AAA' is never true.

The one-dimensional and two-dimensional cases of Postulate 5 are quite dissimilar in substance and might be regarded as separate postulates. If this were done the set Σ_3 would still satisfy all the desired conditions. Independence examples for the one-dimensional case are Examples $\Sigma_1 5$, for the two-dimensional case Examples $\Sigma_1 10$.

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* Postulate $\Sigma_1 14A$ follows from this postulate.

Development.

Theorem 1.* *If* ABC *and* ACD, BCD.

Proof. There exists F such that BFD and CFC (p. 5). $F = C$ (p. 3). $\therefore BCD$.

Theorem 2. *There exist four distinct points.*

Proof. There exist two distinct points A and B (p. 1). There exist points C and D such that $ACB[A \neq C \neq B]$ and $ADC[A \neq D \neq C]$ (p. 2). DCB (th. 1). $\therefore D \neq B$ (p. 3). $\therefore A$, B, C and D are all distinct.

Theorem 3. *If* A *is any point,* AAA.

Proof. There exists B distinct from A (p. 1) and C in the order BCA (p. 2). There exists D in the order BAD (th. 2 and p. $11A$ or p. $11B$). There exists F in the order AFA (p. 5). $F = A$ (p. 3). $\therefore AAA$.

Theorem 4. *If* A *and* B *are any two points,* AAB.

Proof. If $B = A$, AAB (th. 3). If $B \neq A$, there exists C such that ACB and $C \neq B$ (p. 2). $\therefore AAB$ (p. 4).

Theorem 5.† *If* ABC, CBA.

Proof. ABC and AAB (th. 4). Therefore there exists F such that BFB and CFA (p. 5). $F = B$ (p. 3). $\therefore CBA$.

Corollary 1.‡ *If* ABC *and* ABD, *then* ACD *or* ADC *or* $A = B$.

Corollary 2.§ *If* A *and* B *are any two points,* ABB.

Since all the one-dimensional postulates in Σ_1 have now been proved, all the further properties of a line follow as from Σ_1. To establish the categoricity of Σ_3 it only remains to prove the following theorem, from which Postulate $\Sigma_1 10$ follows as a corollary.

If $ABC[A \neq B \neq C]$ and $BDE[B \neq D \neq E]$ and E is not on the line AB, then $C \neq D$ and there exists a point F in the orders CDF and AFE.

Proof from $\Sigma_3 A$. $C \neq D$, as in the proof of Theorem $\Sigma_1 15A$.

Suppose that the lines AE and CD have a point F in common. Then F is in one of the orders $(\alpha)FAE$, $(\beta)AFE$, $(\gamma)AEF$ and in one of the orders (a)FCD, (b)CFD, (c)CDF.

(α). If FAE, there exists (since BDE) a point X in the orders FXD and AXB (p. 5). Therefore X is on the lines AB and $FD (= CD)$. $\therefore X = C$. $\therefore ACB$. But ABC. $\therefore B = C$, contrary to hypothesis.

(b). If CFD, there exists (since CBA) a point X in the orders FXA and DXB. Therefore X is on the lines $AF (= AE)$ and $BD (= BE)$. $\therefore X = E$. $\therefore DEB$. But EDB. $\therefore D = E$, contrary to hypothesis.

(γa). If FEA and FCD, there exists a point X in the orders EXD and

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[*] Postulate $\Sigma_1 5$.

[†] Postulate $\Sigma_1 4$.

[‡] Postulate $\Sigma_1 6$.

[§] Postulate $\Sigma_1 13B$ follows from Postulate $\Sigma_3 11B$ with the aid of this corollary and Theorems 3 and 4.

AXC. Therefore X is on the lines BE and AB. $\therefore X = B$. $\therefore EBD$. But EDB. $\therefore B = D$ contrary to hypothesis.

(γc). If FEA and FDC, there exists a point X in the orders EXC and AXD. Since ABC and AXD, there exists Y in the orders BYD and CYX. Since CYX and CXE, CYE and Y is on the lines CE and BE. $\therefore Y = E$. $\therefore CEX$ and CXE. $\therefore X = E$. $\therefore AED$. But EDB. $\therefore AEB$, contrary to hypothesis.

Since F is in neither of the orders FAE, AEF, it is in the order AFE. Since ABC and AFE, there exists a point X in the orders BXE and CXF. Since X is on both the lines BE and CD, $X = D$. $\therefore CDF$ and the theorem is true if the lines CD and AE have a point in common.

Suppose they have no point in common. There exists a point G distinct from A in the order CAG. Since the distinct lines AB and AE have only A in common, $GE \neq AE$. Moreover GE and AE are both in the plane ABE, in which CD lies. Therefore the lines GE and CD have a point X in common (p. 12A). As before, GXE. Since also GAC, there exists a point F in the orders XFC and EFA. Therefore the lines CD and AE have a point in common, contrary to supposition. The theorem is therefore true.

Proof from $\Sigma_3 B$. $C \neq D$ as before. BAC' and BDE. Therefore there exists a point F in the orders AFE and $C'FD$. Since $C'FD$, CDF.

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